Announcements

1) Fix to 4a)
(need $a \neq 0$) - don't
look at 4b) yet.

2) Map for the course
We'll go through 2.5,
the start on Chapter 4
(determinants), go to
Chapter 5, then back to
Chapter 3.

Proposition: Let V, W, U be vector spaces over $\mathbb{F}$.

Let $T: V \to W$, $S: V \to W$, $R: W \to U$ and $\alpha \in \mathbb{F}$.

If T, S, and R are linear, then

1) αT is linear
2) $T+S$ is linear
3) $R \circ T$ is linear

proof: Let $x, y \in V$, $\beta \in \mathbb{F}$.

1) $(\alpha \cdot T)(x+y)$

$= \alpha \cdot T(x+y)$

(function theoretic)

$= \alpha \cdot (\underbrace{T(x) + T(y)}_{\text{linearity of } T})$

$= \alpha \cdot T(x) + \alpha \cdot T(y)$

$= (\alpha \cdot T)(x) + (\alpha \cdot T)(y)$

$$
\begin{aligned}
(\alpha \cdot T)(\beta \cdot x) &= \alpha \cdot \big(T(\beta \cdot x)\big) \\
&= \alpha \cdot \big(\underbrace{\beta\, T(x)}_{\text{linearity of } T}\big) \\
&= (\alpha \cdot \beta) \cdot T(x) \\
&= (\underbrace{\beta \cdot \alpha}_{\text{commutativity in } \mathbb{F}}) \cdot T(x) \\
&= \beta \cdot \big(\alpha \cdot T(x)\big) \\
&= \beta \big((\alpha \cdot T)(x)\big)
\end{aligned}
$$

2) similarly tedious.

3) $(R \circ T)(x+y)$

$= R(T(x+y))$ (definition)

$$= R(\underbrace{T(x)+T(y)}_{\text{linearity of } T})$$

$$= \underbrace{R(T(x))+R(T(y))}_{\text{linearity of } R}$$

$= (R \circ T)(x) + (R \circ T)(y)$

$$(R \circ T)(\beta x)$$

$$= R(T(\beta x))$$

$$= R(\underbrace{\beta T(x)}_{\text{linearity of } T})$$

$$= \underbrace{\beta \cdot R(T(x))}_{\text{linearity of } R}$$

$$= \beta \cdot ((R \circ T)(x)) \checkmark$$

$$\square$$

Observation: The previous proposition yields that the set of all linear maps between vector spaces V and W (over $\mathbb{F}$) is also a vector space over $\mathbb{F}$, denoted by

$$\mathcal{L}(V, W)$$

or $\mathcal{L}(V)$ when $V = W$

However $\mathcal{L}(V)$ is much more than just a vector space – it is a ring!

In fact, $\mathcal{L}(V)$ is an algebra over $\mathbb{F}$.

Proposition: Let V be a vector space over $\mathbb{F}$. Consider $S, T, R \in \mathcal{L}(V)$.

1) $S \circ (T+R) = S \circ T + S \circ R$

$(S+T) \circ R = S \circ R + T \circ R$

(distributivity)

2) $(S \circ T) \circ R = S \circ (T \circ R)$

(associativity)

3) The map $I_V : V \to V$

given by $I_V(x) = x$

$\forall\, x \in V$ satisfies

$I_V \in \mathcal{L}(V)$ and

$$I_V \circ T = T \circ I_V = T$$

$$\forall\ T \in \mathcal{L}(V)$$

(unit)

4) $\forall\ \alpha \in \mathbb{F}$,

$$\alpha(S \circ T)$$

$$= (\alpha S) \circ T$$

$$= S \circ (\alpha T)$$

(some form of scalar associativity)

proof: 1) only do one of the equalities:

$$(S\circ T + S\circ R)(x)$$

$$= (S\circ T)(x) + (S\circ R)(x)$$

$$= S(T(x)) + S(R(x))$$

$$= \underbrace{S(T(x)+R(x))}_{\text{linearity of } S}$$

$$= S((T+R)(x)) = (S\circ(T+R))(x)$$

2) $((S \circ T) \circ R)(x)$

$= (S \circ T)(R(x))$

$= S(T(R(x)))$

$= S((T \circ R)(x))$

$= (S \circ (T \circ R))(x)$

has nothing to do with linearity!

3) I_V is trivially linear since $\forall\ x, y \in V,\ \alpha \in \mathbb{F}$,

$$I_V(x+y) = x+y$$

$$= I_V(x) + I_V(y)$$

$$I_V(\alpha x) = \alpha x$$

$$= \alpha I_V(x).$$

$$
\begin{aligned}
&(T \circ I_V)(x) \\
&= T(I_V(x)) \\
&= T(x) \\
&= I_V(T(x)) \\
&= (I_V \circ T)(x)
\end{aligned}
$$

and so $I_V \circ T = T = T \circ I_V$.

nothing to do with linearity of T!

4) If $\alpha \in \mathbb{F}$, $x \in V$,

$$\alpha((S \circ T)(x))$$

$$= \alpha(S(T(x)))$$

$$= \underbrace{S(\alpha(T(x)))}_{\text{by linearity of } S}$$

$$= S((\alpha \cdot T)(x))$$

$$= (S \circ (\alpha \cdot T))(x)$$

$$\begin{aligned}
((\alpha S) \circ T)(x) &= ((\alpha \cdot S)(T(x))) \\
&= \alpha \cdot (S(T(x))) \\
&= \alpha \cdot ((S \circ T)(x)) \qquad \square
\end{aligned}$$

Example 1: (noncommutativity)

Consider $V = \mathbb{R}^2$ as a vector space over $\mathbb{R}$.

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $\boxed{T((x,y)) = (3x - 2y, x)}$

and $S: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\boxed{S((x,y)) = (5y, x-y)}$$

$$(T \circ S)((x,y))$$

$$= T((5y, x-y))$$

$$= (15y - 2(x-y), 5y)$$

$$= (17y - 2x, 5y)$$

$$(S \circ T)((x,y))$$

$$= S((3x-2y, x))$$

$$= (5x, 3x-2y-x)$$

$$= (5x, 2x-2y)$$

$\neq (T \circ S)((x,y))$ in general. For example, if $x = y = 1$

$$(S \circ T)((1,1)) = (5,0)$$

$$(T \circ S)((1,1)) = (15,5)$$

So composition need not be commutative useful!